Multiple-Scale Analysis and Renormalization of Quenched Second Order Phase Transitions

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A quenched second order phase transition is modeled by an effective Φ^4-theory with a time-dependent Hamiltonian $\hat{H}(t)$, whose symmetry is broken spontaneously in time. The quantum field evolves out of equilibrium (nonequilibrium) during the phase transition as the density operator significantly deviates from $\hat{\rho}(t) = e^{-\beta \hat{H}(t)}/Z_H$. The recently developed Liouville-von Neumann (LvN) method provides various quantum states for the phase transition in terms of a complex solution to the mean-field equation, which is equivalent to the Gaussian effective potential in the static case and to the time-dependent Hartree-Fock equation in the nonequilibrium case. Using the multiple scale perturbation theory (MSPT) we solve analytically the mean-field equation to the first order of coupling constant and find the quantum states during the quenched second order phase transition. We propose a renormalization scheme during the process of phase transition to regularize the divergences, which originate from the mode coupling between hard and hard modes or between the soft and hard modes. The effect of mode coupling is discussed.

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I. INTRODUCTION

The dynamics of nonequilibrium (out of equilibrium) quantum fields has been a subject of much discussion in the recent years [1,2]. Nonequilibrium quantum fields play important roles in a variety of different scenarios. During a second order phase transition, the time scale of relaxation of the scalar field lags behind the time scale of the change of the effective potential. Consequently the field evolves out of equilibrium as it tries to relax to the new vacuum state, having a nonvanishing vacuum expectation value. Such nonequilibrium effects play a crucial role in topological defect formation both in condensed matter systems as well as in the early universe [3–5]. In a seminal paper, Kibble first showed how the correlation length is crucial in determining the initial density of topological defects [6]. These ideas were applied by Zurek who proposed that it may be possible to quantitatively test the Kibble mechanism of defect formation in condensed matter systems like superfluid He^4 [7]. He argued that, due to the phenomenon of critical slowing down near the phase transition point, the correlation length relevant for determining the initial density of the defects is not the equilibrium correlation length at the Ginzburg temperature but that at the time when the field dynamics essentially freezes out. He also found a power law behaviour in the dependence of the correlation length (and consequently initial defect density) on the quench rate. There have been various attempts to experimentally test the Kibble-Zurek prediction of initial defect density [8]. A comprehensive review of these issues is given in Ref. [2].

The nonequilibrium evolution of a scalar field also is responsible for the establishment of large-scale correlations leading to growth of domains [9]. This is particularly relevant in the context of Disoriented Chiral Condensate (DCC) formation, where only a large domain of DCC can produce the dramatic fluctuations in the charged to neutral pion ratio [10]. The field theoretical treatment from nonequilibrium to equilibrium is particularly important in understanding how a system of quarks and gluons thermalizes after a heavy-ion collision to form a new state of matter called quark-gluon plasma. Searches of quark-gluon plasma are currently underway at RHIC in Brookhaven and LHC in CERN. In another context, the recent preheating mechanism of the Universe after inflation might have accompanied the explosive decay of the inflaton into a large number of soft quanta of the field(s) [11–14]. This process, leading to a large population of the low momentum modes, occurs out of equilibrium. A subsequent interaction between the various modes leads to the transfer of energy from low to high momentum modes and the eventual thermalization of

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the Universe. Another exciting aspect of the nonequilibrium field dynamics which has recently received much attention is the issue of decoherence and consequent appearance of classicality during a nonequilibrium phase transition [15–17].

A feature common to all such (quenched) phase transitions is the evolution far away from equilibrium and the exponential growing of the soft (long wavelength) modes, which necessarily leads to the domain growth. Finite temperature field theory based on equilibrium or quasi-equilibrium methods does not describe all the processes of nonequilibrium evolution even though the imaginary part of the complex effective action yields the decay rate [18]. To treat properly such nonequilibrium quantum evolution, Schwinger and Keldysh first introduced the closed-time formalism [19]. The closed-time formalism or the $1/N$ expansion method have been applied to the nonequilibrium Φ^4-theory to explain the phenomenon of domain growth [9,20]. In a recent paper [21], a canonical method called the Liouville-von Neumann (LvN) approach has been developed that unifies both the functional Schrödinger equation for quantum evolution and the quantum LvN equation for quantum statistics. At the lowest order of coupling parameter the LvN approach yields the Gaussian approximation for static quantum fields and leads to mean-field equations for nonequilibrium quantum fields that can be equivalently obtained by the time-dependent Hartree-Fock method.

The purpose of this paper to study the renormalization of an effective theory of Φ^4 field undergoing a quenched second order phase transition. The Φ may represent a quantum field of an effective system interacting with an environment or an order parameter of the system. In all such cases our model for the second phase transition is described by the Hamiltonian

$$H(t) = \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}} \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m_{\rm B}^2(t)}{2}\Phi^2 + \frac{\lambda_{\rm B}}{4!}\Phi^4 \right], \tag{1}$$

where

$$m_{\rm B}^2(t) = \begin{cases} m_B^2, & t < 0, \\ -\tilde{m}_{\rm B}^2, & t > 0. \end{cases}$$

The transition is modeled through a sudden and instantaneous change in sign of $m_{\rm B}^2$ at $t = 0$; this change destabilizes the field whose true vacuum was at $\Phi = 0$ before $t < 0$ and signifies the onset of the spontaneous symmetry breaking transition. Before the phase transition the quantum field is in a static state of either vacuum state or thermal equilibrium and can be accurately described by the Gaussian effective potential (GEP) [22,23]. But the onset of the quench drives the field out of equilibrium, so the finite temperature field theory or GEP may not be directly applied; instead the LvN or the Hartree-Fock method provides the correct evolution of the field even after the phase transition. In this paper we shall employ the LvN method, as this method exactly recovers the GEP result before the phase transition and the time-dependent Hartree-Fock result after the phase transition.

The Hartree-Fock method has been popular and useful in studying the nonequilibrium evolution of the field [9,20,24]. Even though the renormalization is well understood even for nonequilibrium fields in Refs. [9,20,24], a systematic and explicit renormalization scheme has not been properly addressed for the phase transitions. In this paper we propose such a renormalization scheme for phase transitions by applying the multiple scale perturbation theory (MSPT) method [25,26] to the LvN method and thus obtain the renormalized parameters after phase transitions. The MSPT method introduces the multiple time scales determined by the nonlinear coupling constant of the system, say λ, and leads to the solution with the correct renormalized frequency even at order $\lambda\hbar$. In field theoretical terms the MSPT solution even at the lowest order is equivalent to summing selected diagrams to all orders and is an accurate approximation for a sufficiently long time, $1/(\lambda\hbar)$. The MSPT method turns out particularly useful within the LvN method since the procedure of renormalization in the mean-field or the Hartree-Fock equation amounts to obtaining the renormalized frequencies of the hard and soft modes according to the MSPT method to zeroth order. The calculation is carried out within the Gaussian approximation and, therefore, does not address all the issues pertaining to the non-Gaussian effect of nonlinear mode mixing. It nevertheless provides an analytical (albeit, perturbative) method of obtaining expressions for the renormalized parameters during a spontaneous symmetry breaking phase transition.

The paper is organized as follows. In Sec. II, we show how the LvN method can be used to obtain the renormalized Gaussian effective potential. We also obtain the appropriate form of the renormalized mass and coupling constant using the GEP. In Sec. III, we study a simple quantum-mechanical model of two coupled quartic oscillators as a precursor to studying the nonequilibrium evolution of the quantum fields during a quenched second order phase transition. We apply the MSPT method to obtain the correct frequency for each mode. Although the issue of renormalization is not relevant for this quantum mechanical model, it nevertheless provides some insight into the nonlinear effect of coupling between soft and hard modes during the quenched second order phase transition. In Sec. IV, we deal with the renormalization of mass and coupling parameters during the nonequilibrium evolution of the quantum fields without phase transitions. In Sec. V, by applying the MSPT we obtain the renormalized solution to field equations after the second order phase transition. A summary and discussion of our results is given in Sec. V.

II. GEP FROM THE LVN METHOD

The essential idea of the LvN method [21] is to simultaneously solve the functional Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}(t)|\Psi(t)\rangle, \tag{2}$$

and the quantum LvN equation

$$i\hbar\frac{\partial}{\partial t}\hat{\mathcal{O}}(t) + [\hat{\mathcal{O}}(t), \hat{H}(t)] = 0. \tag{3}$$

The LvN method is based on the observation by Lewis and Riesenfeld [27] that the eigenstates of $\hat{\mathcal{O}}(t)$ are the exact quantum states of time-dependent Schrödinger equation (2) up to time-dependent phase factors. For a time-independent system the Hamiltonian itself satisfies the LvN equation (3) and its energy eigenstates are exact quantum states as expected. In terms of appropriately selected operators satisfying Eq. (3) one may find the density operator and the Hilbert space of exact quantum states. In this sense the LvN method unifies quantum statistical mechanics with quantum mechanics. The LvN method treats the time-dependent, nonequilibrium, system exactly in the same way as the time-independent, equilibrium, one. Also the LvN method can be applied to nonequilibrium fermion systems with a minimal modification [28].

To compare the LvN method with the well-known GEP [23], a nonperturbative method, we consider a field theoretic model to which both methods can apply. Such a model is provided by the Φ^4-theory with the Hamiltonian

$$H(t) = \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}}\left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\Phi)^2 + \frac{m_{\rm B}^2}{2}\Phi^2 + \frac{\lambda_{\rm B}}{4!}\Phi^4\right]. \tag{4}$$

In the GEP method the effective potential of a classical background field ϕ_c includes corrections from quantum fluctuations. The field is assumed to fluctuate in a Gaussian vacuum state with the expectation value

$$\langle 0|\hat{\Phi}|0\rangle_{\rm G} = \phi_c, \quad \langle 0|\hat{\Pi}|0\rangle_{\rm G} = 0, \tag{5}$$

where the field is divided into the classical background field and fluctuations

$$\Phi(t, \mathbf{x}) = \phi_c + \Phi_q(t, \mathbf{x}). \tag{6}$$

To express Eq. (4) as an infinite sum of coupled quartic oscillators, we redefine the Fourier modes of fluctuation as

$$\phi_{\mathbf{k}}^{(+)}(t) = \frac{1}{2}[\phi_{\mathbf{k}}(t) + \phi_{-\mathbf{k}}(t)], \quad \phi_{\mathbf{k}}^{(-)}(t) = \frac{i}{2}[\phi_{\mathbf{k}}(t) - \phi_{-\mathbf{k}}(t)], \tag{7}$$

where

$$\Phi_q(t, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\phi_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{8}$$

Then the mode-decomposed Hamiltonian is given by

$$H = \sum_\alpha \frac{1}{2}\pi_\alpha^2 + \frac{1}{2}\bar{\omega}_\alpha^2\phi_\alpha^2 + \frac{\lambda_{\rm B}}{4!}\left[\sum_\alpha \phi_\alpha^4 + 3\sum_{\alpha\neq\beta}\phi_\alpha^2\phi_\beta^2\right] + \frac{m_{\rm B}^2}{2}\phi_c^2 + \frac{\lambda_{\rm B}}{4!}\phi_c^4 \tag{9}$$

where $\alpha = \{(\pm), \mathbf{k}\}$ and

$$\bar{\omega}_\alpha^2 = m_{\rm B}^2 + \mathbf{k}^2 + \frac{\lambda_{\rm B}}{2}\phi_c^2. \tag{10}$$

Here the odd power terms of ϕ_α are neglected since we are interested in symmetric quantum states. Following Kim and Lee [21], we introduce the operators

$$\hat{A}_\alpha^\dagger(t) = -\frac{i}{\sqrt{\hbar}}[\varphi_\alpha(t)\hat{\pi}_\alpha - \dot{\varphi}_\alpha(t)\hat{\phi}_\alpha], \quad \hat{A}_\alpha(t) = \frac{i}{\sqrt{\hbar}}[\varphi_\alpha^*(t)\hat{\pi}_\alpha - \dot{\varphi}_\alpha^*(t)\hat{\phi}_\alpha], \tag{11}$$

with overdots denoting time derivative, and express the Hamiltonian (9) as normal-ordered quadratic and quartic parts in $\hat{A}$ and $\hat{A}^\dagger$:

$$\hat{H} = \hat{H}_{\rm G} + \hat{H}_{\rm P}, \tag{12}$$

where

$$\begin{aligned}\hat{H}_{\rm G} = \frac{\hbar}{2}\sum_\alpha [(\dot{\varphi}_\alpha^2 + \bar{\omega}_\alpha^2 \varphi^2)\hat{A}_\alpha^2 + (\dot{\varphi}_\alpha^* \dot{\varphi}_\alpha + \bar{\omega}_\alpha^2 \varphi_\alpha^* \varphi_\alpha)(2\hat{A}_\alpha^\dagger \hat{A}_\alpha + 1) + (\dot{\varphi}_\alpha^{*2} + \bar{\omega}_\alpha^2 \varphi_\alpha^{*2})\hat{A}_\alpha^{\dagger 2}] \\ + \frac{\lambda_{\rm B}\hbar^2}{4}\sum_{\alpha,\beta}(\varphi_\beta^*\varphi_\beta)[\varphi_\alpha^2 \hat{A}_\alpha^2 + \varphi_\alpha^*\varphi_\alpha(2\hat{A}_\alpha^\dagger\hat{A}_\alpha + 1) + \varphi_\alpha^{*2}\hat{A}_\alpha^{\dagger 2}] + \frac{m_{\rm B}^2}{2}\phi_c^2 + \frac{\lambda_{\rm B}}{4!}\phi_c^4 \\ - \frac{\lambda_{\rm B}\hbar^2}{8}\left(\sum_\alpha \varphi_\alpha^*\varphi_\alpha\right)^2,\end{aligned} \tag{13}$$

and

$$\begin{aligned}\hat{H}_{\rm P} = \frac{\lambda_{\rm B}\hbar^2}{4!}\Bigg\{\sum_\alpha \sum_{k=0}^{4}\binom{4}{k}\varphi_\alpha^{*(4-k)}\varphi_\alpha^k \hat{A}_\alpha^{\dagger(4-k)}\hat{A}_\alpha^k \\ + \sum_{\alpha\neq\beta} 3[(\dot{\varphi}_\alpha^2 + \bar{\omega}_\alpha^2\varphi_\alpha^2)\hat{A}_\alpha^2 + 2(\dot{\varphi}_\alpha^*\dot{\varphi}_\alpha + \omega_\alpha^2\varphi_\alpha^*\varphi_\alpha)\hat{A}_\alpha^\dagger\hat{A}_\alpha + (\dot{\varphi}_\alpha^{*2} + \bar{\omega}_\alpha^2\varphi_\alpha^{*2})\hat{A}_\alpha^{\dagger 2}] \\ \times[\dot{\varphi}_\beta^2 + \omega_\beta^2 v_\beta^2)\hat{A}_\beta^2 + 2(\dot{\varphi}_\beta^*\dot{\varphi}_\beta + \bar{\omega}_\beta^2 v_\beta^* v_\beta)\hat{A}_\beta^\dagger\hat{A}_\beta + (\dot{\varphi}_\beta^{*2} + \bar{\omega}_\beta^2\varphi_\beta^{*2})\hat{A}_\beta^{\dagger 2}]\Bigg\}.\end{aligned} \tag{14}$$

The strategy of the GEP and LvN methods is to exactly solve the quadratic part $\hat{H}_{\rm G}$ and perturbatively treat the quartic part $\hat{H}_{\rm P}$. In the GEP method, the GEP is obtained by minimizing the free energy with respect to a variable frequency, which consequently determines φ_α. In the LvN method, the requirement that the creation and annihilation operators (11) satisfy the LvN equation (3) leads to the mean-field equations for φ_α.

In the GEP method for a static system with time-independent coupling parameters [22,23], one minimizes the vacuum energy with respect to a trial Gaussian state of the form

$$\Psi_{\Omega_\alpha}(\phi_\alpha) = \left(\frac{\Omega_\alpha}{\pi}\right)^{1/4} e^{-\frac{i}{2}\Omega_\alpha t}\exp\left[-\frac{\Omega_\alpha}{2\hbar}\phi_\alpha^2\right], \tag{15}$$

where Ω_α are free parameters. The variational parameters Ω_α are time-independent for the static system with time-independent coupling parameters, but depend on time for a nonequilibrium system with time-dependent coupling parameters. The time-dependent variational principle by Dirac [29] is required for the time-dependent system, to which we shall apply the LvN method in this paper. The time-dependent phase factor in Eq. (15) has been inserted to satisfy the time-dependent Schrödinger equation. Indeed, the Gaussian state (15) is the vacuum state annihilated by $\hat{A}$ in Eq. (11), which can be shown by choosing the solution

$$\varphi_\alpha = \frac{1}{\sqrt{2\Omega_\alpha}}e^{-i\Omega_\alpha t}, \tag{16}$$

and writing Eq. (11) in the canonical form

$$\hat{A}_\alpha^\dagger(t) = e^{-i\Omega_\alpha t}\left[\sqrt{\frac{\Omega_\alpha}{2\hbar}}\hat{\phi}_\alpha - i\sqrt{\frac{1}{2\hbar\Omega_\alpha}}\hat{\pi}_\alpha\right], \quad \hat{A}_\alpha(t) = e^{i\Omega_\alpha t}\left[\sqrt{\frac{\Omega_\alpha}{2\hbar}}\hat{\phi}_\alpha + i\sqrt{\frac{1}{2\hbar\Omega_\alpha}}\hat{\pi}_\alpha\right]. \tag{17}$$

Then the Hamiltonian leads to the Gaussian effective potential

$$V_{\rm G} = \langle \hat{H}\rangle_{\rm G} = \frac{\hbar}{4}\sum_\alpha\left(\Omega_\alpha + \frac{\bar{\omega}_\alpha^2}{\Omega_\alpha}\right) + \frac{\lambda_{\rm B}\hbar^2}{8}\left(\sum_\alpha \frac{1}{2\Omega_\alpha}\right)^2 + \frac{m_{\rm B}^2}{2}\phi_c^2 + \frac{\lambda_{\rm B}}{4!}\phi_c^4. \tag{18}$$

The minimization of the Gaussian effective potential with respect to Ω_α leads to the gap equation

$$
\begin{aligned}
\Omega_\alpha^2 &= \bar{\omega}_\alpha^2 + \frac{\lambda_{\rm B}\hbar}{2}\sum_\beta \frac{1}{2\Omega_\beta} \\
&= \mathbf{k}^2 + \mu^2(\phi_c),
\end{aligned} \tag{19}
$$

where

$$
\mu^2(\phi_c) = m_{\rm B}^2 + \frac{\lambda_{\rm B}}{2}\phi_c^2 + \frac{\lambda_{\rm B}\hbar}{2}\sum_\beta \frac{1}{2\Omega_\beta}. \tag{20}
$$

Now the Gaussian effective potential takes the form

$$
V_{\rm G} = \frac{m_{\rm B}^2}{2}\phi_c^2 + \frac{\lambda_{\rm B}}{4!}\phi_c^4 + \hbar I_1(\mu) - \frac{\lambda_{\rm B}\hbar^2}{8} I_0^2(\mu), \tag{21}
$$

where

$$
I_n(\mu) = \sum_\alpha \frac{1}{2\Omega_\alpha^{1-2n}} = \frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\Omega_{\mathbf{k}}^{1-2n}}. \tag{22}
$$

In the GEP method the renormalization of coupling parameters (constants) are prescribed by matching the coefficients of the effective potential with renormalized ones. Following Ref. [23], the renormalized mass in the Gaussian state, a symmetric state around $\phi_c = 0$, is given by

$$
\begin{aligned}
m_{\rm R}^2 &= \left.\frac{d^2 V_{\rm G}}{d\phi_c^2}\right|_{\phi_c=0} \\
&= m_{\rm B}^2 + \frac{\lambda_{\rm B}\hbar}{2} I_0(\mu_0).
\end{aligned} \tag{23}
$$

Similarly, the renormalized coupling constant is found to be given by

$$
\begin{aligned}
\lambda_{\rm R} &= \left.\frac{d^4 V_{\rm G}}{d\phi_c^4}\right|_{\phi_c=0} \\
&= \lambda_{\rm B}\frac{[1 - \frac{\lambda_{\rm B}\hbar}{2} I_{-1}(\mu)]}{[1 + \frac{\lambda_{\rm B}\hbar}{4} I_{-1}(\mu)]}.
\end{aligned} \tag{24}
$$

The case of particularly relevance to our model with a spontaneous symmetry breaking is when $\lambda_{\rm B}$ is positive and finite [23]. The other cases corresponding to different values of $\lambda_{\rm B}$ have also been discussed in detail [23]. In our model the time-dependent mass term spontaneously breaks the symmetry during the quenched phase transition, which may be regarded as a consequence of the interaction with an environment.

On the other hand, in the LvN method, the operators (11) are required to satisfy the LvN equation (3) for the truncated Hamiltonian $\hat{H}_{\rm G}$. This leads to the mean-field equation for each mode

$$
\ddot{\varphi}_\alpha + \left[\bar{\omega}_\alpha^2 + \frac{\lambda_{\rm B}\hbar}{2}\sum_\beta (\varphi_\beta^*\varphi_\beta)\right]\varphi_\alpha = 0. \tag{25}
$$

The LvN method allows even time-dependent coupling parameters as long as Eq. (25) is satisfied. Further, the operators (11) are endowed with the standard commutators

$$
[\hat{A}_\alpha(t), \hat{A}_\beta^\dagger(t)] = \delta_{\alpha,\beta}, \tag{26}
$$

which are guaranteed by the Wronskian conditions

$$
\dot{\varphi}_\alpha^*(t)\varphi_\alpha(t) - \varphi_\alpha^*(t)\dot{\varphi}_\alpha(t) = i. \tag{27}
$$

The Fock space for each mode is constructed from the time-dependent creation and annihilation operators, $\hat{A}_\alpha^\dagger(t)$ and $\hat{A}_\alpha(t)$. The Gaussian vacuum state of αth mode, which is an approximation to the true vacuum state, is annihilated by $\hat{A}_\alpha(t)$:

$$\hat{A}_\alpha(t)|0_\alpha, t\rangle_{\rm G} = 0, \tag{28}$$

and has the coordinate representation

$$\Psi_{0_\alpha}(\phi_\alpha, t) = \frac{1}{(2\pi\hbar\varphi_\alpha^*\varphi_\alpha)^{1/4}}\left(\frac{\varphi_\alpha}{\varphi_\alpha^*}\right)^{1/2}\exp\left[\frac{i}{2\hbar}\frac{\dot{\varphi}_\alpha^*}{\varphi_\alpha^*}\phi_\alpha^2\right]. \tag{29}$$

Similarly, the nth excited state is obtained by applying $\hat{A}^\dagger_\alpha(t)$ n times:

$$|n_\alpha, t\rangle = \frac{1}{\sqrt{n_\alpha!}}\left(\hat{A}^\dagger_\alpha(t)\right)^{n_\alpha}|0_\alpha, t\rangle, \tag{30}$$

and, in the coordinate representation, is given by

$$\Psi_{n_\alpha}(\phi_\alpha, t) = \frac{1}{((2\hbar)^n n!)^{1/2}}\frac{1}{(2\pi\hbar\varphi_\alpha^*\varphi_\alpha)^{1/4}}\left(\frac{\varphi_\alpha}{\varphi_\alpha^*}\right)^{n+1/2} H_n\left(\frac{\phi_\alpha^2}{\sqrt{2\hbar\varphi_\alpha^*\varphi_\alpha}}\right)\exp\left[\frac{i}{2\hbar}\frac{\dot{\varphi}_\alpha^*}{\varphi_\alpha^*}\phi_\alpha^2\right], \tag{31}$$

where H_n is the Hermite polynomial.

A few comments are in order. The quantum states (29) and (31) in general depend on time, except for trivial time-dependent phase factors, whenever the coupling parameter $m_{\rm B}$ or $\lambda_{\rm B}$ depends on time. In fact, these states constitute the time-dependent Fock space. The Gaussian vacuum state of the field Φ is the product of Gaussian state of each mode:

$$|0, t\rangle_{\rm G} = \prod_\alpha |0_\alpha, t\rangle_{\rm G}. \tag{32}$$

Similarly, the excited state of the field is the product of each mode state, at least one state being excited. This Fock representation of quantum field is unitarily inequivalent because all excited states of the field are not countable [30]. Further, there is another feature of the nonequilibrium field that is absent from the static quantum field. In spite of the fact that each Fock space at two different times is unitary equivalent through a Bogoliubov transformation, the Gaussian vacuum states of the field are orthogonal to each other at two different times due to infinite number of modes:

$$_{\rm G}\langle 0, t'|0, t\rangle_{\rm G} = 0, \quad (t' \neq t). \tag{33}$$

Now the task in the LvN method is to solve the mean-field equation (25). For the static system the complex function (16) is a solution to Eq. (25) if Ω_α satisfies the gap equation (19). This shows the equivalence between the LvN and GEP methods for at least static case. Also it follows that the mean-field equations (25) for symmetric quantum states with $\phi_c = 0$, upon mass renormalization (23), become renormalized

$$\ddot{\varphi}_{{\rm R}\alpha} + (m_{\rm R}^2 + \mathbf{k}^2)\varphi_{{\rm R}\alpha} = 0, \tag{34}$$

where the renormalized frequency is

$$\begin{aligned}\Omega_{{\rm R}\alpha}^2 &= \Omega_\alpha^2\Big|_{\phi_c=0} \\ &= \mathbf{k}^2 + m_{\rm R}^2.\end{aligned} \tag{35}$$

Therefore, a simple renormalized solution is given by

$$\varphi_{{\rm R}\alpha} = \frac{1}{\sqrt{2\Omega_{{\rm R}\alpha}}}e^{-i\Omega_{{\rm R}\alpha}t}. \tag{36}$$

This implies that the renormalized Gaussian vacuum state (29) and its excited states (31) for the static Hamiltonian (4) indeed describe the free theory with the renormalized mass.

We now compare the Gaussian effective potential (18) with the conventional perturbation theory. This comparison provides a direct interpretation of the results of the MSPT method, which coincide with the results of GEP method to first order of $\lambda_{\rm B}\hbar$, as will be shown in Secs. IV and V. The renormalized mass is particularly of interest from the point of view of instability during the phase transition. The conventional perturbation theory is based on the solution

$$\varphi_\alpha^{(0)} = \frac{1}{\sqrt{2\bar{\omega}_\alpha}} e^{-i\bar{\omega}_\alpha t}. \tag{37}$$

It is a solution to Eq. (25) without the self-interaction term ($\lambda_{\rm B} = 0$) and leads to a different Gaussian vacuum state (29), which is denoted as $|0, t\rangle_{(0)}$. So one obtains another effective potential

$$V_{(0)} = \langle \hat{H} \rangle_{(0)} = \frac{\hbar}{2} \sum_\alpha \bar{\omega}_\alpha + \frac{\lambda_{\rm B} \hbar^2}{8} \left(\sum_\alpha \frac{1}{2\bar{\omega}_\alpha} \right)^2 + \frac{m_{\rm B}^2}{2} \phi_c^2 + \frac{\lambda_{\rm B}}{4!} \phi_c^4. \tag{38}$$

The renormalized mass to order of $(\lambda_{\rm B} \hbar)^2$ is given by

$$\begin{aligned} m_{{\rm R}(0)}^2 &= \left. \frac{d^2 V_{(0)}}{d\phi_c^2} \right|_{\phi_c = 0} \\ &= m_{\rm B}^2 + \frac{\lambda_{\rm B} \hbar}{2} J_0 - \frac{(\lambda_{\rm B} \hbar)^2}{8} J_0 J_{-1}, \end{aligned} \tag{39}$$

where

$$\begin{aligned} J_n &= \sum_\alpha \frac{1}{2\omega_\alpha^{1-2n}} = \int \frac{d^3 \mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\omega_{\mathbf{k}}^{1-2n}}, \\ \omega_\alpha^2 &= m_{\rm B}^2 + \mathbf{k}^2. \end{aligned} \tag{40}$$

To compare Eq. (39) with the GEP results, (18) and (19), we first expand Eq. (19) as

$$\Omega_\alpha(\mu_0) = \omega_\alpha + \frac{\lambda_{\rm B} \hbar}{4\omega_\alpha} - \frac{(\lambda_{\rm B} \hbar)^2}{8\omega_\alpha} J_0 J_{-1} - \frac{(\lambda_{\rm B} \hbar)^2}{32 \omega_\alpha^3} J_0^2 + \mathcal{O}(\lambda_{\rm B} \hbar)^3, \tag{41}$$

and then obtain

$$\begin{aligned} I_0(\mu_0) &= J_0 - \frac{\lambda_{\rm B} \hbar}{4} J_0 J_{-1} + \mathcal{O}(\lambda_{\rm B} \hbar)^2, \\ I_{-1}(\mu_0) &= J_{-1} - \frac{3\lambda_{\rm B} \hbar}{4} J_0 J_{-2} + \mathcal{O}(\lambda_{\rm B} \hbar)^2. \end{aligned} \tag{42}$$

Hence, the renormalized mass (23)

$$m_{\rm R}^2 = m_{\rm B}^2 + \frac{\lambda_{\rm B} \hbar}{2} J_0 - \frac{(\lambda_{\rm B} \hbar)^2}{8} J_0 J_{-1} + \mathcal{O}(\lambda_{\rm B} \hbar)^3, \tag{43}$$

coincides with Eq. (39) up to $(\lambda \hbar)^2$. All higher order terms in the expansion of $m_{\rm R}^2$ and $\lambda_{\rm R}$ are a consequence of summing over daisy and superdaisy diagrams in GEP.

Finally, we compare the LvN method with the time-dependent Hartree-Fock method. We divide the field and the momentum into a classical background field and a quantum fluctuation:

$$\Phi(t, \mathbf{X}) = \phi_c + \Phi_q(t, \mathbf{x}), \quad \Pi(t, \mathbf{X}) = \pi_c + \Pi_q(t, \mathbf{x}). \tag{44}$$

The Gaussian state has not only $\langle \hat{\Phi} \rangle = \phi_c$ but also $\langle \hat{\Pi} \rangle = \pi_c$. The quantum fluctuations have the zero vacuum expectation value, $\langle \hat{\Phi}_q \rangle = 0$ and $\langle \hat{\Pi}_q \rangle = 0$. In the Hartree-Fock method, $\hat{\Phi}_q^3$ is replaced by $3\langle \hat{\Phi}_q^2 \rangle \hat{\Phi}_q$. Then the field equation for quantum fluctuation is given by

$$\partial_\mu \partial^\mu \hat{\Phi}_q + \left(m_{\rm B}^2 + \frac{\lambda_{\rm B}}{2} \phi_c^2 + \frac{\lambda_{\rm B}}{2} \langle \hat{\Phi}_q^2 \rangle \right) \hat{\Phi}_q = 0, \tag{45}$$

and for classical field by

$$\partial_\mu \partial^\mu \phi_c + \left(m_{\rm B}^2 + \frac{\lambda_{\rm B}}{2} \langle \hat{\Phi}_q^2 \rangle \right) \phi_c + \frac{\lambda_{\rm B}}{3!} \phi_c^3 = 0. \tag{46}$$

To remove the infinite term

$$\langle\hat{\Phi}_q^2\rangle = \hbar\sum_\alpha \varphi_\alpha^*\varphi_\alpha = \hbar I_0(\mu_0), \tag{47}$$

we add counter terms $(\delta m^2/2)\Phi^2$ and $(\delta\lambda/4!)\Phi^4$, which is equivalent to writing the bare coupling parameters as

$$m_{\rm B}^2 = m_{\rm R}^2 + \delta m^2, \quad \lambda_{\rm B} = \lambda_{\rm R} + \delta\lambda. \tag{48}$$

By choosing the counter mass term as $\delta m^2 = -(\lambda_{\rm B}\hbar/2)I_0(\mu_0)$, we obtain from Eq. (45) the renormalized field equation for $\phi_c = 0$

$$(\partial_\mu\partial^\mu + m_{\rm R}^2)\hat{\Phi}_{{\rm R}q} = 0. \tag{49}$$

Note that the renormalized mass obtained by adding the mass counter term is the same as Eq. (23) in the GEP or the LvN method and the Fourier transform of Eq. (49) would be the operator counterpart of the renormalized mean-field equation (34).

III. ANALYSIS OF THE COUPLED QUARTIC OSCILLATOR MODEL USING MSPT METHOD

The multiple scale perturbation theory (MSPT) was introduced to find solutions to nonlinear systems with a perturbatively small coupling constant, say $\lambda\hbar$ [25,26]. The nonlinear systems exhibit distinct characteristic behaviour on different time scales determined by $\lambda\hbar$. A conventional perturbation theory leads to secular (boundlessly growing) terms due to the resonant coupling between a lower order and the next leading order. However, the MSPT method provides a systematic technique of eliminating the secular terms and, as a consequence, gives rise to the solution with an effective frequency, which can be obtained by summing the most secular terms to all orders of $\lambda\hbar$ in the conventional perturbation theory [26]. Even the lowest order MSPT solution is valid over a time scale $1/(\lambda\hbar)$. In this section we apply the MSPT method to analyze the system of two coupled quartic oscillators. It is a useful toy model for getting some insight into the phenomenon of mode mixing that one encounters in the study of nonequilibrium quantum fields during a spontaneous symmetry breaking phase transition. The MSPT was applied to a single classical as well as a quantum anharmonic oscillator in Ref. [26]. In the latter case, the technique involves solving a complicated set of coupled operator equations. However, in the LvN method, the auxiliary equations for the Gaussian state or its excited states are the classical mean-field type equations or the linearized Hartree-Fock equations, which can be more easily tackled. As mentioned earlier (see also [21]), these coupled equations are a direct consequence of the fact that the creation and annihilation operators given in Eq. (11) satisfy the LvN equation.

The model to be studied in this section is described by the Hamiltonian

$$H = \frac{p_1^2}{2} + \frac{\omega_1^2}{2}q_1^2 + \frac{p_2^2}{2} + \frac{\omega_2^2}{2}q_2^2 + \frac{\lambda}{4!}\left(q_1^4 + 6q_1^2q_2^2 + q_2^4\right). \tag{50}$$

The Hamiltonian (50) is obtained by taking two typical modes of scalar field from the field Hamiltonian (9). The most physically interesting case is the coupling between the hard and soft modes. The approximate Gaussian state for the Hamiltonian (50) with $\langle\hat{q}_1\rangle = 0 = \langle\hat{q}_2\rangle$ is given by the product

$$\Psi_0(q_1, q_2) = \Psi_1(q_1)\Psi_2(q_2) \tag{51}$$

of the Gaussian state for each mode:

$$\Psi_i(q_i) = \frac{1}{(2\pi\hbar v_i^* v_i)^{1/4}}\left(\frac{v_i}{v_i^*}\right)^{1/2} \exp\left[\frac{i}{2\hbar}\frac{\dot{v}_i^*}{v_i^*}q_i^2\right]. \tag{52}$$

The $v_i, (i = 1, 2)$, satisfy the coupled equations

$$\ddot{v}_1 + \omega_1^2 v_1 + \frac{\lambda\hbar}{2}(v_1^* v_1 + v_2^* v_2)v_1 = 0,$$
$$\ddot{v}_2 + \omega_2^2 v_2 + \frac{\lambda\hbar}{2}(v_1^* v_1 + v_2^* v_2)v_2 = 0, \tag{53}$$

together with the Wronskian conditions

$$\dot{v}_i^* v_i - \dot{v}_i v_i^* = i. \tag{54}$$

Note that $(\lambda\hbar)$ in Eq. (53) is the nonlinear coupling constant. As each time scale of order $(\lambda\hbar)^n t \sim \mathcal{O}(1)$ with $n = 1, 2, \cdots$, shows a different characteristic behavior of the solution, we introduce a multiple of different time scales

$$\tau_{(1)} = (\lambda\hbar)t, \cdots, \tau_{(n)} = (\lambda\hbar)^n t, \cdots. \tag{55}$$

The solution to Eq. (53) is expanded in a series of $\lambda\hbar$ as

$$v_i = \sum_{n=0} (\lambda\hbar)^n v_i^{(n)}(t, \tau_{(1)}, \cdots, \tau_{(n)}, \cdots), \tag{56}$$

where $v_i^{(n)}$ is of the order of unity. The time derivative is now given by

$$\begin{aligned}
\frac{\partial^2 v_i}{\partial t^2} &= \frac{\partial^2 v_i^{(0)}}{\partial t^2} + 2\frac{\partial^2 v_i^{(0)}}{\partial t \partial \tau_{(1)}}\left(\frac{\partial \tau_{(1)}}{\partial t}\right) + \frac{\partial^2 v_i^{(0)}}{\partial \tau_{(1)}^2}\left(\frac{\partial \tau_{(1)}}{\partial t}\right)^2 + 2\frac{\partial^2 v_i^{(0)}}{\partial t \partial \tau_{(2)}}\left(\frac{\partial \tau_{(2)}}{\partial t}\right) + \cdots \\
&+\lambda\hbar\left\{\frac{\partial^2 v_i^{(1)}}{\partial t^2} + 2\frac{\partial^2 v_i^{(1)}}{\partial t \partial \tau_{(1)}}\left(\frac{\partial \tau_{(1)}}{\partial t}\right) + \frac{\partial^2 v_i^{(1)}}{\partial \tau_{(1)}^2}\left(\frac{\partial \tau_{(1)}}{\partial t}\right)^2 + \cdots\right\} \\
&+(\lambda\hbar)^2\left\{\frac{\partial^2 v_i^{(2)}}{\partial t^2} + 2\frac{\partial^2 v_i^{(2)}}{\partial t \partial \tau_{(1)}}\left(\frac{\partial \tau_{(1)}}{\partial t}\right) + \frac{\partial^2 v_i^{(2)}}{\partial \tau_{(1)}^2}\left(\frac{\partial \tau_{(1)}}{\partial t}\right)^2 + \cdots\right\} + \cdots
\end{aligned} \tag{57}$$

To zeroth order $(\lambda\hbar)^0$, $v_i^{(0)}$ satisfies the simple oscillator equation

$$\ddot{v}_i^{(0)} + \omega_i^2 v_i^{(0)} = 0. \tag{58}$$

First, we consider the case $\omega_i^2 \geq 0$, which is the quantum mechanical analog of the coupling between two hard modes. As $\tau \equiv \tau_{(1)}$ is a time scale independent of t, one looks for the zeroth order solution to Eq. (58) of the form

$$v_i^{(0)} = A_i(\tau)e^{i\omega_i t} + B_i(\tau)e^{-i\omega_i t}. \tag{59}$$

Collecting terms of order $(\lambda\hbar)$ from Eq. (57), one obtains the nonlinear equation

$$\left\{\frac{\partial^2 v_i^{(1)}}{\partial t^2} + \omega_i^2 v_i^{(1)}\right\} + \left\{2\frac{\partial^2 v_i^{(0)}}{\partial t \partial \tau} + \frac{1}{2}\left(v_1^{(0)*}v_1^{(0)} + v_2^{(0)*}v_2^{(0)}\right)v_i^{(0)}\right\} = 0. \tag{60}$$

The secular terms, which are proportional to $e^{\pm i\omega_i t}$ in the second curly bracket of Eq. (60), should be eliminated to prevent $v_i^{(1)}$ from growing boundlessly in time. This requires the coefficients to satisfy

$$\begin{aligned}
\frac{\partial A_i}{\partial \tau} &= \frac{i}{4\omega_i}\left\{B_i^* B_i + \sum_{j=1,2} A_j^* A_j + B_j^* B_j\right\}A_i, \\
\frac{\partial B_i}{\partial \tau} &= \frac{-i}{4\omega_i}\left\{A_i^* A_i + \sum_{j=1,2} A_j^* A_j + B_j^* B_j\right\}B_i.
\end{aligned} \tag{61}$$

Equation (61) together with its complex conjugate for the coefficients $A_i^*(\tau_{(1)})$ and $B_i^*(\tau_{(1)})$, determines the initial values

$$A_i^*(\tau)A_i(\tau) = A_i^*(0)A_i(0), \quad B_i^*(\tau)B_i(\tau) = B_i^*(0)B_i(0). \tag{62}$$

Now the solutions to Eq. (61) are given by

$$\begin{aligned}
A_i(\tau) &= A_i(0)\exp\left[\frac{i\tau}{4\omega_i}\left\{B_i^*(0)B_i(0) + \sum_{j=1,2}(A_j^*(0)A_j(0) + B_j^*(0)B_j(0))\right\}\right], \\
B_i(\tau) &= B_i(0)\exp\left[\frac{-i\tau}{4\omega_i}\left\{A_i^*(0)A_i(0) + \sum_{j=1,2}(A_j^*(0)A_j(0) + B_j^*(0)B_j(0))\right\}\right].
\end{aligned} \tag{63}$$

At order $(\lambda\hbar)^0$, the initial Gaussian state is given by

$$A_i(0) = 0, \quad B_i(0) = \frac{1}{\sqrt{2\omega_i}}. \tag{64}$$

Therefore, the solution for $v_i^{(0)}$ becomes

$$v_i^{(0)} = \frac{1}{\sqrt{2\omega_i}} e^{-i\Omega_i^{(-)} t}, \tag{65}$$

where each mode has a new shifted frequency

$$\Omega_i^{(-)} = \omega_i + \frac{\lambda\hbar}{4\omega_i} \sum_{j=1,2} \frac{1}{2\omega_j}. \tag{66}$$

The second term can be obtained by summing the most secular terms to all orders in the conventional perturbation theory. As each $v_i^{(0)}$ satisfies the linear equation (58), we choose the coefficient to satisfy the Wronskian condition (54)

$$v_i = \frac{1}{\sqrt{2\Omega_i^{(-)}}} e^{-i\Omega_i^{(-)} t}. \tag{67}$$

Note that the solution (67), when expanded in power of $\lambda\hbar$, yields the first order solution $v_i^{(1)}$, as it has the correct frequency and coefficient up to that order. By comparing Eq. (66) with the gap equation

$$\Omega_i = \left[\omega_i^2 + \frac{\lambda\hbar}{2} \sum_{j=1,2} \frac{1}{2\Omega_j}\right]^{1/2}, \tag{68}$$

obtained from the exact solution to Eq. (53)

$$v_i = \frac{1}{\sqrt{2\Omega_i}} e^{-i\Omega_i t}, \tag{69}$$

it can be shown that $\Omega_i^{(-)}$ is the binomial expansion of Ω_i to order of $\lambda\hbar$. Hence the lowest order MSPT solution gives the correct frequency or the energy to order $(\lambda\hbar)$.

We now consider the more interesting case of one hard and one soft mode. To mimic the interaction between the hard and soft modes after a sudden quench, one frequency squared, say ω_2^2, will instantaneously change the sign from positive to negative at $t = 0$, i.e., $\omega_1^2 \to \tilde{\omega}_1^2$ and $\omega_2^2 \to -\tilde{\omega}_2^2$. Then Eq. (50) for $t > 0$ takes the form

$$\tilde{H} = \frac{p_1^2}{2} + \frac{\tilde{\omega}_1^2}{2} q_1^2 + \frac{p_2^2}{2} - \frac{\tilde{\omega}_2^2}{2} q_2^2 + \frac{\lambda}{4!}\left(q_1^4 + 6q_1^2 q_2^2 + q_2^4\right), \tag{70}$$

exactly the Hamiltonian for the interaction between a hard mode and a soft mode in Φ^4-theory which will be discussed in the next section. The equations for the auxiliary variables $\tilde{v}_1$ and $\tilde{v}_2$ to $\mathcal{O}((\lambda\hbar)^0)$ become

$$\begin{aligned} \ddot{\tilde{v}}_1^{(0)} + \tilde{\omega}_1^2 \tilde{v}_1^{(0)} &= 0, \\ \ddot{\tilde{v}}_2^{(0)} - \tilde{\omega}_2^2 \tilde{v}_2^{(0)} &= 0. \end{aligned} \tag{71}$$

As for the former case of two hard modes, $\tau = (\lambda\hbar)t$ introduces another time scale in addition to t, and the solutions to Eq. (71) take the form

$$\begin{aligned} \tilde{v}_1^{(0)} &= \tilde{A}(\tau) e^{i\tilde{\omega}_1 t} + \tilde{B}(\tau) e^{-i\tilde{\omega}_1 t}, \\ \tilde{v}_2^{(0)} &= \tilde{C}(\tau) e^{\tilde{\omega}_2 t} + \tilde{D}(\tau) e^{-\tilde{\omega}_2 t}. \end{aligned} \tag{72}$$

Secular terms appear in equations for $\tilde{v}_i^{(1)}$ due to the resonant coupling with $\tilde{v}_i^{(0)}$. These are the terms proportional to $e^{\pm i\tilde{\omega}_1 t}$ for $\tilde{v}_1^{(1)}$ and $e^{\pm \tilde{\omega}_2 t}$ for $\tilde{v}_2^{(2)}$ and lead to boundlessly growing solutions. By eliminating these terms, one obtains the equations

$$
\begin{aligned}
\frac{\partial \tilde{A}}{\partial \tau} &= \frac{i}{4\tilde{\omega}_1}\left\{\tilde{A}^*\tilde{A} + 2\tilde{B}^*\tilde{B} + \tilde{C}^*\tilde{D} + \tilde{C}\tilde{D}^*\right\}\tilde{A}, \\
\frac{\partial \tilde{B}}{\partial \tau} &= \frac{-i}{4\tilde{\omega}_1}\left\{2\tilde{A}^*\tilde{A} + \tilde{B}^*\tilde{B} + \tilde{C}^*\tilde{D} + \tilde{C}\tilde{D}^*\right\}\tilde{B}, \\
\frac{\partial \tilde{C}}{\partial \tau} &= \frac{-1}{4\tilde{\omega}_2}\left\{\tilde{A}^*\tilde{A} + \tilde{B}^*\tilde{B} + 2\tilde{C}^*\tilde{D} + \tilde{C}\tilde{D}^*\right\}\tilde{C}, \\
\frac{\partial \tilde{D}}{\partial \tau} &= \frac{1}{4\tilde{\omega}_2}\left\{\tilde{A}^*\tilde{A} + \tilde{B}^*\tilde{B} + \tilde{C}^*\tilde{D} + 2\tilde{C}\tilde{D}^*\right\}\tilde{D}.
\end{aligned} \tag{73}
$$

A similar set of equations involving coefficients $\tilde{A}^*$, $\tilde{B}^*$, $\tilde{C}^*$, $\tilde{D}^*$, is obtained by requiring that the secular terms for $\tilde{v}_1^{(1)*}$ and $\tilde{v}_2^{(1)*}$ vanish, which is the complex conjugate of Eq. (73). Then it follows that

$$
\begin{aligned}
\tilde{A}^*(\tau)\tilde{A}(\tau) = \tilde{A}^*(0)\tilde{A}(0), \quad \tilde{B}^*(\tau)\tilde{B}(\tau) = \tilde{B}^*(0)\tilde{B}(0), \\
\tilde{C}^*(\tau)\tilde{D}(\tau) = \tilde{C}^*(0)\tilde{D}(0), \quad \tilde{C}(\tau)\tilde{D}^*(\tau) = \tilde{C}(0)\tilde{D}^*(0).
\end{aligned} \tag{74}
$$

As the system evolves from an initial Gaussian state for $t < 0$ to another Gaussian state for $t > 0$ after the phase transition, the Gaussian state (52) should be continuous across $t = 0$. The continuity of the Gaussian state is equivalent to the continuity of v_i and $\tilde{v}_i$ at $t = 0$, which determines the coefficients of the zeroth order solutions after the phase transition in terms of the initial data:

$$
\begin{aligned}
\tilde{A}(0) &= \frac{1}{2\sqrt{2\omega_1}}\left(1 - \frac{\omega_1}{\tilde{\omega}_1}\right), \quad \tilde{B}(0) = \frac{1}{2\sqrt{2\omega_1}}\left(1 + \frac{\omega_1}{\tilde{\omega}_1}\right), \\
\tilde{C}_2(0) &= \frac{1}{2\sqrt{2\omega_2}}\left(1 - i\frac{\omega_2}{\tilde{\omega}_2}\right), \quad \tilde{D}_2(0) = \frac{1}{2\sqrt{2\omega_2}}\left(1 + i\frac{\omega_2}{\tilde{\omega}_2}\right).
\end{aligned} \tag{75}
$$

By substituting Eq. (75) into Eq. (74) and solving Eq. (73), one finally obtains

$$
\begin{aligned}
\tilde{v}_1^{(0)}(t) &= \tilde{A}(0)e^{i\tilde{\Omega}_1^{(+)}t} + \tilde{B}(0)e^{-i\tilde{\Omega}_1^{(-)}t}, \\
\tilde{v}_2^{(0)}(t) &= \tilde{C}(0)e^{\tilde{\Omega}_2^{(+)}t} + \tilde{D}(0)e^{-\tilde{\Omega}_2^{(-)}t},
\end{aligned} \tag{76}
$$

where

$$
\begin{aligned}
\tilde{\Omega}_1^{(\pm)} &= \tilde{\omega}_1 + \frac{\lambda\hbar}{4\tilde{\omega}_1}\left\{\frac{3}{8\omega_1}\left(1 + \frac{\omega_1^2}{\tilde{\omega}_1^2}\right) + \frac{1}{4\omega_2}\left(1 - \frac{\omega_2^2}{\tilde{\omega}_2^2}\right) \pm \frac{1}{4\tilde{\omega}_1}\right\}, \\
\tilde{\Omega}_2^{(\pm)} &= \tilde{\omega}_2 - \frac{\lambda\hbar}{4\tilde{\omega}_2}\left\{\frac{1}{4\omega_1}\left(1 + \frac{\omega_1^2}{\tilde{\omega}_1^2}\right) + \frac{3}{8\omega_2}\left(1 - \frac{\omega_2^2}{\tilde{\omega}_2^2}\right) \pm \frac{i}{4\tilde{\omega}_2}\right\}.
\end{aligned} \tag{77}
$$

By requiring the Wronskian conditions (54) and matching the first order solutions (67), we find the correct first order coefficients as

$$
\begin{aligned}
\tilde{v}_1(t) &= \frac{1}{\sqrt{2\Omega_1^{(-)}}}\left(\frac{\tilde{\Omega}_1^{(-)} - \Omega_1^{(-)}}{\tilde{\Omega}_1^{(+)} + \tilde{\Omega}_1^{(-)}}\right)e^{i\tilde{\Omega}_1^{(+)}t} + \frac{1}{\sqrt{2\Omega_1^{(-)}}}\left(\frac{\tilde{\Omega}_1^{(+)} + \Omega_1^{(-)}}{\tilde{\Omega}_1^{(+)} + \tilde{\Omega}_1^{(-)}}\right)e^{-i\tilde{\Omega}_1^{(-)}t}, \\
\tilde{v}_2(t) &= \frac{1}{\sqrt{2\Omega_2^{(-)}}}\left(\frac{\tilde{\Omega}_1^{(-)} - i\Omega_1^{(-)}}{\tilde{\Omega}_1^{(+)} + \tilde{\Omega}_1^{(-)}}\right)e^{\tilde{\Omega}_2^{(+)}t} + \frac{1}{\sqrt{2\Omega_2^{(-)}}}\left(\frac{\tilde{\Omega}_1^{(+)} + i\Omega_1^{(-)}}{\tilde{\Omega}_1^{(+)} + \tilde{\Omega}_1^{(-)}}\right)e^{-\tilde{\Omega}_2^{(-)}t}.
\end{aligned} \tag{78}
$$

A few comments on the interpretation of solutions (65) and (78) are in order. These solutions are valid till $t \sim 1/(\lambda\hbar)$. The coupling between the hard modes just modifies the frequencies (66) even after the phase transition. However, the coupling between the hard and soft modes after the phase transition exhibits some interesting features for the both modes. One interesting feature is the disparity between the positive and negative frequency given by

$\delta\tilde{\Omega}_1 \equiv \tilde{\Omega}_1^{(+)} - \tilde{\Omega}_1^{(-)} = \lambda/8\tilde{\omega}_1^2$ for the hard mode and $\delta\tilde{\Omega}_2 \equiv \tilde{\Omega}_2^{(+)} - \tilde{\Omega}_2^{(-)} = -i\lambda/8\tilde{\omega}_2^2$ for the soft mode. This difference is a consequence of the resonant coupling for the positive and the negative frequency solution and for the exponentially growing and decaying solutions. Another interesting feature is the appearance of an imaginary part of $\tilde{\Omega}_2^{(\pm)}$ for the soft mode. This implies that $\tilde{v}_2$ begins to oscillate due to the coupling between the hard and hard modes. The time scale of oscillation is estimated to be $(16\omega_2^2)/(\lambda\hbar)$, which is larger by the factor $16\omega_2^2$ than the time scale $1/\lambda\hbar$ for the lowest order solution to be valid. The period of the oscillation is too long to be observed.

Though the time scale for oscillation is outside the validity region of the zeroth order solution, the oscillatory behavior gives more information about the later stage of evolution towards and oscillation around the true vacuum. The numerical analysis of the first order solution shows that the early stage of exponential growth due to instability is eventually terminated by the back-reaction from nonlinear terms [31]. It is the oscillatory part that determines the absolute width of the Gaussian state after the field samples the true vacuum. To see how the oscillatory part to restrict the Gaussian state, we use the lowest order solution $\tilde{v}_2$, which is the auxiliary equation for the Gaussian state for the soft mode. The correct physical interpretation may follow from the Gaussian state (52):

$$\tilde{\Psi}_2(q_2, t) = \frac{1}{(2\pi\hbar\tilde{v}_2^*\tilde{v}_2)^{1/4}} \left(\frac{\tilde{v}_2}{\tilde{v}_2^*}\right)^{1/2} \exp\Biggl[-\Biggl\{\frac{\lambda}{32\tilde{\omega}_2^2} + \frac{2\omega_2}{\hbar\tilde{\omega}_2}\mathrm{Re}(\tilde{\omega}_2^{(+)})e^{-2\mathrm{Re}(\tilde{\omega}_2^{(+)})t} + \cdots\Biggr\}q_2^2 \\ + i\{\frac{\mathrm{Re}(\tilde{\omega}_2^{(+)})}{2\hbar}(1 - e^{-4\mathrm{Re}(\tilde{\omega}_2^{(+)})t} + \cdots)\}q_2^2\Biggr]. \tag{79}$$

Immediately after the phase transition, the width (dispersion) of the Gaussian state (79) is largely determined by the second term of the real part which originates from the exponentially growing part of $\tilde{v}_2$. But, as the phase transition continues, the first term from the imaginary part of $\tilde{\Omega}_2^{(\pm)}$ is comparable to the second term when

$$e^{-2\mathrm{Re}(\tilde{\omega}_2^{(+)})t} \approx \frac{\lambda\hbar}{64\tilde{\omega}_2^3}. \tag{80}$$

The time scale of Eq. (80) is outside the validity of the lowest order solution. Before reaching the time (80), the back-reaction $(\lambda\hbar/2)(\tilde{v}_2^*\tilde{v}_2)\tilde{v}_2$ of the self-interaction begins to dominate over $-\tilde{\omega}_2^2\tilde{v}_2$. So the more accurate first order solution oscillates around the true vacuum after an exponentially growing period and the Gaussian state has approximately a constant width, like the first term of the real part. That is, the Gaussian state stops spreading.

IV. MSPT AND RENORMALIZED FREQUENCY

The main advantages of the MSPT method, especially within the framework of the LvN formalism, was discussed in the previous section. Nevertheless, it is necessary to make some comments on this method in the field theoretic context. The process of eliminating the secular terms at various orders of the MSPT method is equivalent to summing certain diagrams to all orders in the conventional perturbation theory. It is precisely because of this fact that useful information pertaining to the divergence structure of the theory can be extracted even from the lowest order solution in the MSPT method. As will be shown below, the lowest order MSPT solution of a quantum field yields a shifted (effective) frequency which contains divergent contributions and the removal of the most divergent term essentially amounts to mass renormalization.

The requirement that the creation and annihilation operators (11) satisfy the LvN equation, leads to the mean-field equation of the auxiliary field variable of each mode

$$\ddot{\varphi}_\alpha + \omega_\alpha^2\varphi_\alpha + \frac{\lambda_{\mathrm{B}}\hbar}{2}\left(\sum_\beta \varphi_\beta^*\varphi_\beta\right)\varphi_\alpha = 0, \tag{81}$$

where

$$\omega_\alpha^2 = m_{\mathrm{B}}^2 + \mathbf{k}^2. \tag{82}$$

The exact solution to Eq. (81) for the static system without phase transition is given by Eq. (16). Now we apply the MSPT to solve perturbatively Eq. (81) in a power series of $\lambda_{\mathrm{B}}\hbar$ and to find the correct renormalized frequency. As in Sec. III, introduce a multiple of time scales

$$\tau_{(1)} = (\lambda_{\mathrm{B}}\hbar)t, \cdots, \tau_{(n)} = (\lambda_{\mathrm{B}}\hbar)^n t, \cdots, \tag{83}$$

and expand the solution in the power series of $\lambda\hbar$

$$\varphi_\alpha = \sum_{n=0} (\lambda_{\rm B}\hbar)^n \varphi_\alpha^{(n)}(t, \tau_{(1)}, \cdots, \tau_{(n)}, \cdots). \tag{84}$$

As we shall find the solution (84) only to the first order, we denote $\tau_{(1)} = \tau$ and truncate it to that order

$$\varphi_\alpha(t) = \varphi_\alpha^{(0)}(t) + (\lambda_{\rm B}\hbar)\varphi_\alpha^{(1)}(t,\tau). \tag{85}$$

The time derivative is now given by

$$\frac{\partial^2 \varphi_\alpha(t)}{\partial t^2} = \frac{\partial^2 \varphi_\alpha^{(0)}}{\partial t^2} + 2\frac{\partial^2 \varphi_\alpha^{(0)}(t)}{\partial t \partial \tau}\left(\frac{\partial \tau}{\partial t}\right) + (\lambda_{\rm B}\hbar)\frac{\partial^2 \varphi_\alpha^{(1)}}{\partial t^2} + \cdots. \tag{86}$$

The terms to order $(\lambda_{\rm B}\hbar)^0$ satisfy the equation for a simple oscillator

$$\ddot{\varphi}_\alpha^{(0)} + \omega_\alpha^2 \varphi_\alpha^{(0)} = 0. \tag{87}$$

The general solution is a superposition of the positive and negative frequency solutions

$$\varphi_\alpha^{(0)} = A_\alpha(\tau)e^{i\omega_\alpha t} + B_\alpha(\tau)e^{-i\omega_\alpha t}. \tag{88}$$

At the next order the terms of order $(\lambda_{\rm B}\hbar)$ in Eq. (81) obey the equation

$$\left\{\frac{\partial^2 \varphi_\alpha^{(1)}}{\partial t^2} + \omega_\alpha^2 \varphi_\alpha^{(1)}\right\} + \left\{2\frac{\partial^2 \varphi_\alpha^{(0)}}{\partial t \partial \tau} + \frac{1}{2}\left(\sum_\beta \varphi_\beta^{(0)*}\varphi_\beta^{(0)}\right)\varphi_\alpha^{(0)}\right\} = 0. \tag{89}$$

As $\varphi_\alpha^{(1)}$ has the same natural frequency ω_α as $\varphi_\alpha^{(0)}$, any term proportional to $e^{\pm i\omega_\alpha t}$ in the second curly bracket is a secular term to $\varphi_\alpha^{(1)}$. To avoid unnecessary growing terms, these secular terms should be eliminated from the second curly bracket. This requires

$$\begin{aligned}
\frac{\partial A_\alpha}{\partial \tau} &= \frac{i}{4\omega_\alpha}\left\{B_\alpha^* B_\alpha + \sum_\beta A_\beta^* A_\beta + B_\beta^* B_\beta\right\}A_\alpha, \\
\frac{\partial B_\alpha}{\partial \tau} &= \frac{-i}{4\omega_\alpha}\left\{A_\alpha^* A_\alpha + \sum_\beta A_\beta^* A_\beta + B_\beta^* B_\beta\right\}B_\alpha.
\end{aligned} \tag{90}$$

As the coefficients of A_α and B_α are real, Eq. (90) is unitary, so has the solution

$$A_\alpha^*(\tau)A_\alpha(\tau) = A_\alpha^*(0)A_\alpha(0), \quad B_\alpha^*(\tau)B_\alpha(\tau) = B_\alpha^*(0)B_\alpha(0). \tag{91}$$

Hence the solutions are given by

$$\begin{aligned}
A_\alpha(\tau) &= A_\alpha(0)\exp\left[\frac{i\tau}{4\omega_\alpha}\left\{B_\alpha^*(0)B_\alpha(0) + \sum_\beta A_\beta^*(0)A_\beta(0) + B_\beta^*(0)B_\beta(0)\right\}\right], \\
B_\alpha(\tau) &= B_\alpha(0)\exp\left[\frac{-i\tau}{4\omega_\alpha}\left\{A_\alpha^*(0)A_\alpha(0) + \sum_\beta A_\beta^*(0)A_\beta(0) + B_\beta^*(0)B_\beta(0)\right\}\right].
\end{aligned} \tag{92}$$

As the Gaussian state before the phase transition has the initial data

$$A_\alpha(0) = 0, \quad B_\alpha(0) = \frac{1}{\sqrt{2\omega_\alpha}}, \tag{93}$$

the solution becomes

$$\varphi_\alpha^{(0)} = \frac{1}{\sqrt{2\omega_\alpha}}\exp[-i\Omega_\alpha^{(-)}t], \tag{94}$$

where

$$\Omega_\alpha^{(-)} = \omega_\alpha + \frac{\lambda_{\rm B}\hbar}{4\omega_\alpha}\sum_\alpha \frac{1}{2\omega_{\mathbf{k}}} = \omega_\alpha + \frac{\lambda_{\rm B}\hbar}{4\omega_\alpha} J_0. \tag{95}$$

By requiring the solution (94) to satisfy the Wronskian condition (27), the solution is given by

$$\varphi_\alpha = \frac{1}{\sqrt{2\Omega_\alpha^{(-)}}} \exp[-i\Omega_\alpha^{(-)} t]. \tag{96}$$

The solution (96) of the MSPT method coincides with the frequency (41) to the first order of $\lambda_{\rm B}\hbar$. As a consequence, the MSPT solution (96) recovers the GEP (18) and renormalized mass (23) to the first order $\lambda_{\rm B}\hbar$. All higher order corrections to the MSPT solution are expected to exactly recover the solution (16) of GEP. In this sense the MSPT method can be understood as a powerful tool to systematically solve Eq. (25) even for the nonequilibrium case with time-dependent coupling parameters.

We now apply the MSPT to a quantum field changing suddenly from one parameter $m_{\rm B}^2$ to another $\tilde{m}_{\rm B}^2 \geq 0$ at $t = 0$. The mean field equation (81) still applies with the modification

$$\tilde{\omega}_\alpha^2 = \tilde{m}_{\rm B}^2 + \mathbf{k}^2, \tag{97}$$

and the solution (88) has the form

$$\tilde{\varphi}_\alpha^{(0)}(t) = \tilde{A}_\alpha(\tau) e^{i\tilde{\omega}_\alpha t} + \tilde{B}_\alpha(\tau) e^{-i\tilde{\omega}_\alpha t}. \tag{98}$$

The solution is given by

$$\begin{aligned} \tilde{A}_\alpha(\tau) &= \tilde{A}_\alpha(0) \exp\left[\frac{i\tau}{4\tilde{\omega}_\alpha}\left\{\tilde{B}_\alpha^*(0)\tilde{B}_\alpha(0) + \sum_\beta \tilde{A}_\beta^*(0)\tilde{A}_\beta(0) + \tilde{B}_\beta^*(0)\tilde{B}_\beta(0)\right\}\right], \\ \tilde{B}_\alpha(\tau) &= \tilde{B}_\alpha(0) \exp\left[\frac{-i\tau}{4\tilde{\omega}_\alpha}\left\{\tilde{A}_\alpha^*(0)\tilde{A}_\alpha(0) + \sum_\beta \tilde{A}_\beta^*(0)\tilde{A}_\beta(0) + \tilde{B}_\beta^*(0)\tilde{B}_\beta(0)\right\}\right]. \end{aligned} \tag{99}$$

The initial data is determined by the continuity of φ_α and $\tilde{\varphi}_\alpha$ at $t = 0$:

$$\tilde{A}_\alpha(0) = \frac{1}{2\sqrt{2\omega_\alpha}}\left(1 - \frac{\omega_\alpha}{\tilde{\omega}_\alpha}\right), \quad \tilde{B}_\alpha(0) = \frac{1}{2\sqrt{2\omega_\alpha}}\left(1 + \frac{\omega_\alpha}{\tilde{\omega}_\alpha}\right). \tag{100}$$

The zeroth order solution can then written be in terms of the modified frequencies as

$$\tilde{\varphi}_\alpha^{(0)}(\tau) = \tilde{A}_\alpha(0) \exp(i\tilde{\Omega}_\alpha^{(+)} t) + \tilde{B}_\alpha(0) \exp[-i\tilde{\Omega}_\alpha^{(-)} t]. \tag{101}$$

where

$$\tilde{\Omega}_\alpha^{(\pm)} = \tilde{\omega}_\alpha + \frac{\lambda_{\rm B}\hbar}{4\tilde{\omega}_\alpha}\left[\sum_\beta \frac{1}{2\omega_\beta} + \frac{(m_{\rm B}^2 - \tilde{m}_{\rm B}^2)}{2}\sum_\beta \frac{1}{2\omega_\beta\tilde{\omega}_\beta^2} + \frac{1}{8\omega_\alpha}\left(1 \pm \frac{\omega_\alpha}{\tilde{\omega}_\alpha}\right)^2\right]. \tag{102}$$

The shifted frequency (102) can be written as

$$\tilde{\Omega}_\alpha^{(\pm)} = \tilde{\omega}_\alpha + \frac{\lambda_{\rm B}\hbar}{4\tilde{\omega}_\alpha}\left[J_0 + \frac{(m_{\rm B}^2 - \tilde{m}_{\rm B}^2)}{2} J_{-1} + \frac{(m_{\rm R}^2 - \tilde{m}_{\rm R}^2)^2}{2}\sum_\beta \frac{1}{2\omega_\beta^3\tilde{\omega}_\beta^2} + \frac{1}{8\omega_\alpha}\left(1 \pm \frac{\omega_\alpha}{\tilde{\omega}_\alpha}\right)^2\right], \tag{103}$$

where we used the relation

$$m_{\rm B}^2 - \tilde{m}_{\rm B}^2 = m_{\rm R}^2 - \tilde{m}_{\rm R}^2 \tag{104}$$

and rewrote the sum

$$\sum_\beta \frac{1}{2\omega_\beta \tilde{\omega}_\beta^2} = \sum_\beta \frac{1}{2\omega_\beta^3} + \sum_\beta \frac{\omega_\beta^2 - \tilde{\omega}_\beta^2}{2\omega_\beta^3 \tilde{\omega}_\beta^2}$$
$$= J_{-1} + (m_{\rm R}^2 - \tilde{m}_{\rm R}^2) \sum_\beta \frac{1}{2\omega_\beta^3 \tilde{\omega}_\beta^2}. \tag{105}$$

Finally, by fixing the coefficients of Eq. (101) to match Eq. (96) and satisfy Eq. (27) we obtain the lowest order solutions

$$\tilde{\varphi}_\alpha(\tau) = \frac{1}{\sqrt{2\Omega_\alpha^{(-)}}} \left(\frac{\tilde{\Omega}_\alpha^{(-)} - \Omega_\alpha^{(-)}}{\tilde{\Omega}_\alpha^{(+)} + \tilde{\Omega}_\alpha^{(-)}} \right) \exp[i\tilde{\Omega}_\alpha^{(+)} t] + \frac{1}{\sqrt{2\Omega_\alpha^{(-)}}} \left(\frac{\tilde{\Omega}_\alpha^{(+)} + \Omega_\alpha^{(-)}}{\tilde{\Omega}_\alpha^{(+)} + \tilde{\Omega}_\alpha^{(-)}} \right) \exp[-i\tilde{\Omega}_\alpha^{(-)} t]. \tag{106}$$

From the GEP calculation, Eqs. (19) and (35), of Sec. II , it follows that the new renormalized frequencies can be written as

$$\tilde{\Omega}_\alpha^{(\pm)} \simeq \Omega_{{\rm R}\alpha} + \frac{\lambda_{\rm B}\hbar}{4\tilde{\omega}_\alpha} \left[\frac{(m_{\rm B}^2 - \tilde{m}_{\rm B}^2)}{2} J_{-1} + \frac{(m_{\rm R}^2 - \tilde{m}_{\rm R}^2)^2}{2} \sum_\beta \frac{1}{2\omega_\beta^3 \tilde{\omega}_\beta^2} + \frac{1}{8\omega_\alpha} \left(1 \pm \frac{\omega_\alpha}{\tilde{\omega}_\alpha} \right)^2 \right]. \tag{107}$$

V. RENORMALIZED SOLUTION AFTER PHASE TRANSITION

After the quenched phase transition at $t = 0$, the soft modes (long wavelength) with momenta $k < \tilde{m}_{\rm B}$ become unstable and evolve out of equilibrium, whereas the hard modes (short wavelength) with $k > \tilde{m}_{\rm B}$ evolve towards a new equilibrium. The Gaussian state after phase transition can then be determined by two sets of the auxiliary fields, one for the stable hard modes and the other for the unstable soft modes. The mean field equation (81) for the hard modes then become

$$\ddot{\tilde{\varphi}}_{{\rm H}\alpha} + \tilde{\omega}_{{\rm H}\alpha}^2 \tilde{\varphi}_{{\rm H}\alpha} + \frac{\lambda_{\rm B}\hbar}{2} \left(\sum_\beta \tilde{\varphi}_\beta^* \tilde{\varphi}_\beta \right) \tilde{\varphi}_{{\rm H}\alpha} = 0, \tag{108}$$

where

$$\tilde{\omega}_{{\rm H}\alpha}^2 = \mathbf{k}^2 - \tilde{m}_{\rm B}^2, \tag{109}$$

and for the soft modes

$$\ddot{\tilde{\varphi}}_{{\rm S}\alpha} - \tilde{\omega}_{{\rm S}\alpha}^2 \tilde{\varphi}_{{\rm S}\alpha} + \frac{\lambda_{\rm B}\hbar}{2} \left(\sum_\beta \tilde{\varphi}_\beta^* \tilde{\varphi}_\beta \right) \tilde{\varphi}_{{\rm S}\alpha} = 0, \tag{110}$$

where

$$\tilde{\omega}_{{\rm S}\alpha}^2 = \tilde{m}_{\rm B}^2 - \mathbf{k}^2. \tag{111}$$

Once again, by introducing two independent time scales t and τ in accordance with the MSPT method, the solutions of zero order $\mathcal{O}((\lambda_B\hbar)^0)$ for the hard and soft modes can be written as

$$\tilde{\varphi}_{{\rm H}\alpha}^{(0)} = \tilde{A}_\alpha(\tau) e^{i\tilde{\omega}_{{\rm H}\alpha} t} + \tilde{B}_\alpha(\tau) e^{-i\tilde{\omega}_{{\rm H}\alpha} t}, \tag{112}$$
$$\tilde{\varphi}_{{\rm S}\alpha}^{(0)} = \tilde{C}_\alpha(\tau) e^{\tilde{\omega}_{{\rm S}\alpha} t} + \tilde{D}_\alpha(\tau) e^{-\tilde{\omega}_{{\rm S}\alpha} t}. \tag{113}$$

Substituting these zeroth order solutions into Eqs. (108) and (110) and eliminating the secular terms $e^{\pm i\tilde{\omega}_{{\rm H}\alpha} t}$ for $\tilde{\varphi}_{{\rm H}\alpha}^{(1)}$ and $e^{\pm\tilde{\omega}_{{\rm S}\alpha} t}$ for $\tilde{\varphi}_{{\rm S}\alpha}^{(1)}$, one finds the equations for coefficients:

$$
\begin{aligned}
\frac{\partial \tilde{A}_\alpha}{\partial \tau} &= \frac{i}{4\tilde{\omega}_{\mathrm{H}\alpha}}\left\{\tilde{B}^*_\alpha \tilde{B}_\alpha + \sum_{\beta_>} \tilde{A}^*_\beta \tilde{A}_\beta + \tilde{B}^*_\beta \tilde{B}_\beta + \sum_{\beta_<} \tilde{C}^*_\beta \tilde{D}_\beta + \tilde{C}_\beta \tilde{D}^*_\beta\right\}\tilde{A}_\alpha, \\
\frac{\partial \tilde{B}_\alpha}{\partial \tau} &= \frac{-i}{4\omega_{\mathrm{H}\alpha}}\left\{\tilde{A}^*_\alpha \tilde{A}_\alpha + \sum_{\beta_>} \tilde{A}^*_\beta \tilde{A}_\beta + \tilde{B}^*_\beta \tilde{B}_\beta + \sum_{\beta_<} \tilde{C}^*_\beta \tilde{D}_\beta + \tilde{C}_\beta \tilde{D}^*_\beta\right\}\tilde{B}_\alpha, \\
\frac{\partial \tilde{C}_\alpha}{\partial \tau} &= \frac{-1}{4\omega_{\mathrm{S}\alpha}}\left\{\tilde{C}^*_\alpha \tilde{D}_\alpha + \sum_{\beta_>} \tilde{A}^*_\beta \tilde{A}_\beta + \tilde{B}^*_\beta \tilde{B}_\beta + \sum_{\beta_<} \tilde{C}^*_\beta \tilde{D}_\beta + \tilde{C}_\beta \tilde{D}^*_\beta\right\}\tilde{C}_\alpha, \\
\frac{\partial \tilde{D}_\alpha}{\partial \tau} &= \frac{1}{4\omega_{\mathrm{S}\alpha}}\left\{\tilde{D}^*_\alpha \tilde{C}_\alpha + \sum_{\beta_>} \tilde{A}^*_\beta \tilde{A}_\beta + \tilde{B}^*_\beta \tilde{B}_\beta + \sum_{\beta_<} \tilde{C}^*_\beta \tilde{D}_\beta + \tilde{C}_\beta \tilde{D}^*_\beta\right\}\tilde{D}_\alpha. \qquad (114)
\end{aligned}
$$

Here $\beta_>$ and $\beta_<$ denote the restricted sum (integral) over the hard and soft modes, respectively. Another set of equations, the complex conjugate of Eq. (114), can be obtained for the coefficients $\tilde{A}^*_\alpha$, $\tilde{B}^*_\alpha$, $\tilde{C}^*_\alpha$ and $\tilde{D}^*_\alpha$ by eliminating the secular terms for $\tilde{\varphi}^{*(1)}_{\mathrm{H}\alpha}$ and $\tilde{\varphi}^{*(1)}_{\mathrm{S}\alpha}$. This set of equations together with Eq. (114) determines the initial values of the coefficients:

$$
\begin{aligned}
&\tilde{A}^*_\alpha(\tau)\tilde{A}_\alpha(\tau) = \tilde{A}^*_\alpha(0)\tilde{A}_\alpha(0), \quad \tilde{B}^*_\alpha(\tau)\tilde{B}_\alpha(\tau) = \tilde{B}^*_\alpha(0)\tilde{B}_\alpha(0), \\
&\tilde{C}^*_\alpha(\tau)\tilde{D}_\alpha(\tau) = \tilde{C}^*_\alpha(0)\tilde{D}_\alpha(0), \quad \tilde{C}_\alpha(\tau)\tilde{D}^*_\alpha(\tau) = \tilde{C}_\alpha(0)\tilde{D}^*_\alpha(0). \qquad (115)
\end{aligned}
$$

By using the above relations, we find the solution of Eq. (114) as

$$
\begin{aligned}
\tilde{A}_\alpha(\tau) &= \tilde{A}_\alpha(0)\exp\left[\frac{i\tau}{4\omega_{\mathrm{H}\alpha}}\left\{\tilde{B}^*_\alpha \tilde{B}_\alpha + \sum_{\beta_>} \tilde{A}^*_\beta \tilde{A}_\beta + \tilde{B}^*_\beta \tilde{B}_\beta + \sum_{\beta_<} \tilde{C}^*_\beta \tilde{D}_\beta + \tilde{C}_\beta \tilde{D}^*_\beta\right\}\right], \\
\tilde{B}_\alpha(\tau) &= \tilde{B}_\alpha(0)\exp\left[\frac{-i\tau}{4\omega_{\mathrm{H}\alpha}}\left\{\tilde{A}^*_\alpha \tilde{A}_\alpha + \sum_{\beta_>} \tilde{A}^*_\beta \tilde{A}_\beta + \tilde{B}^*_\beta \tilde{B}_\beta + \sum_{\beta_<} \tilde{C}^*_\beta \tilde{D}_\beta + \tilde{C}_\beta \tilde{D}^*_\beta\right\}\right], \\
\tilde{C}_\alpha(\tau) &= \tilde{C}_\alpha(0)\exp\left[\frac{-\tau}{4\omega_{\mathrm{S}\alpha}}\left\{\tilde{C}^*_\alpha \tilde{D}_\alpha + \sum_{\beta_>} \tilde{A}^*_\beta \tilde{A}_\beta + \tilde{B}^*_\beta \tilde{B}_\beta + \sum_{\beta_<} \tilde{C}^*_\beta \tilde{D}_\beta + \tilde{C}_\beta \tilde{D}^*_\beta\right\}\right], \\
\tilde{D}_\alpha(\tau) &= \tilde{D}_\alpha(0)\exp\left[\frac{\tau}{4\omega_{\mathrm{S}\alpha}}\left\{\tilde{D}^*_\alpha \tilde{C}_\alpha + \sum_{\beta_>} \tilde{A}^*_\beta \tilde{A}_\beta + \tilde{B}^*_\beta \tilde{B}_\beta + \sum_{\beta_<} \tilde{C}^*_\beta \tilde{D}_\beta + \tilde{C}_\beta \tilde{D}^*_\beta\right\}\right]. \qquad (116)
\end{aligned}
$$

The initial data (115) are determined by continuity of $\varphi^{(0)}_\alpha$ and $\tilde{\varphi}^{(0)}_\alpha$:

$$
\begin{aligned}
\tilde{A}_\alpha(0) &= \frac{1}{2\sqrt{2\omega_\alpha}}\left(1 - \frac{\omega_\alpha}{\tilde{\omega}_{\mathrm{H}\alpha}}\right), \quad \tilde{B}_\alpha(0) = \frac{1}{2\sqrt{2\omega_\alpha}}\left(1 + \frac{\omega_\alpha}{\tilde{\omega}_{\mathrm{H}\alpha}}\right), \\
\tilde{C}_\alpha(0) &= \frac{1}{2\sqrt{2\omega_\alpha}}\left(1 - i\frac{\omega_\alpha}{\tilde{\omega}_{\mathrm{S}\alpha}}\right), \quad \tilde{D}_\alpha(0) = \frac{1}{2\sqrt{2\omega_\alpha}}\left(1 + i\frac{\omega_\alpha}{\tilde{\omega}_{\mathrm{S}\alpha}}\right). \qquad (117)
\end{aligned}
$$

It is worth noting that the integral for the soft modes, the last two terms of the summation in the exponent of Eq. (116), is a restricted one with its upper limit being $\tilde{m}_{\mathrm{B}}$. On the other hand, the integral for the soft modes, the first two terms of the summation in the exponent of Eq. (116), has $\tilde{m}_{\mathrm{B}}$ as its lower limit of integration. For $k \gg \tilde{m}_{\mathrm{B}}$, one can write $\tilde{\omega}_{\mathrm{H}\beta} \sim \omega_\beta$, whence the ultraviolet (quadratic) divergence mainly comes from $\sum_\beta \tilde{B}^*_\beta \tilde{B}_\beta$ for the hard modes. In terms of new frequencies $\tilde{\Omega}^{(\pm)}_{\mathrm{H}\alpha}$ for the hard modes and $\tilde{\Omega}^{(\pm)}_{\mathrm{S}\alpha}$ for the soft modes, the zeroth order solution can be written as

$$
\begin{aligned}
\tilde{\varphi}^{(0)}_{\mathrm{H}\alpha} &= \tilde{A}_\alpha(0)e^{i\tilde{\Omega}^{(+)}_{\mathrm{H}\alpha}t} + \tilde{B}_\alpha(0)e^{-i\tilde{\Omega}^{(-)}_{\mathrm{H}\alpha}t}, \\
\tilde{\varphi}^{(0)}_{\mathrm{S}\alpha} &= \tilde{C}_\alpha(0)e^{\tilde{\Omega}^{(+)}_{\mathrm{S}\alpha}t} + \tilde{D}_\alpha(0)e^{\tilde{\Omega}^{(-)}_{\mathrm{S}\alpha}t} \qquad (118)
\end{aligned}
$$

where

$$\tilde{\Omega}^{(\pm)}_{\mathrm{H}\alpha} = \tilde{\omega}_{\mathrm{H}\alpha} + \frac{\lambda_{\mathrm{B}}\hbar}{4\tilde{\omega}_{\mathrm{H}\alpha}}\left[\sum_{\beta}\frac{1}{2\omega_\beta} + \frac{(m^2_{\mathrm{B}}+\tilde{m}^2_{\mathrm{B}})}{2}\left\{\sum_{\beta_>}\frac{1}{2\omega_\beta\tilde{\omega}^2_{\mathrm{H}\beta}} - \sum_{\beta_<}\frac{1}{2\omega_\beta\tilde{\omega}^2_{\mathrm{S}\beta}}\right\} + \frac{1}{8\omega_\alpha}\left(1 \pm \frac{\omega_\alpha}{\tilde{\omega}_{\mathrm{H}\alpha}}\right)^2\right],$$
$$\tilde{\Omega}^{(\pm)}_{\mathrm{S}\alpha} = \tilde{\omega}_{\mathrm{S}\alpha} - \frac{\lambda_{\mathrm{B}}\hbar}{4\tilde{\omega}_{\mathrm{S}\alpha}}\left[\sum_{\beta}\frac{1}{2\omega_\beta} + \frac{(m^2_{\mathrm{B}}+\tilde{m}^2_{\mathrm{B}})}{2}\left\{\sum_{\beta_>}\frac{1}{2\omega_\beta\tilde{\omega}^2_{\mathrm{H}\beta}} - \sum_{\beta_<}\frac{1}{2\omega_\beta\tilde{\omega}^2_{\mathrm{S}\beta}}\right\} + \frac{1}{8\omega_\alpha}\left(1 \pm i\frac{\omega_\alpha}{\tilde{\omega}_{\mathrm{S}\alpha}}\right)^2\right]. \quad (119)$$

Here $\sum_{\beta_<}$ denotes the restricted integral over soft modes, having a finite upper momentum cut-off given by $\tilde{m}_{\mathrm{B}}$ and, therefore, making a finite contribution to the renormalized frequencies. However, the quadratic ultraviolet divergent term has to be appropriately regularized to yield the renormalized frequencies. Contrary to a naive belief that the frequency of the soft modes may be finite, the soft frequency also has the same ultraviolet divergence as the hard modes and renormalization is required to remove it. This is a consequence of the nonlinear coupling between the soft and the hard modes. As the mass counter term δm^2 has the same sign, $\tilde{m}^2_{\mathrm{B}} = \tilde{m}^2_{\mathrm{R}} - \delta m^2$, even after the phase transition, the sum of bare squared masses after the phase transition equals to that of renormalized squared masses:

$$m^2_{\mathrm{B}} + \tilde{m}^2_{\mathrm{B}} = m^2_{\mathrm{R}} + \tilde{m}^2_{\mathrm{R}}. \quad (120)$$

Using the identity

$$\sum_{\beta_>}\frac{1}{2\omega_\beta\tilde{\omega}^2_{\mathrm{H}\beta}} = \sum_{\beta_>}\frac{1}{2\omega^3_\beta} + (m^2_{\mathrm{R}}+\tilde{m}^2_{\mathrm{R}})\sum_{\beta_>}\frac{1}{2\omega^3_\beta\tilde{\omega}^2_{\mathrm{H}\beta}},$$
$$\sum_{\beta<}\frac{1}{2\omega_\beta\tilde{\omega}^2_{\mathrm{S}\beta}} = -\sum_{\beta_<}\frac{1}{2\omega^3_\beta} + (m^2_{\mathrm{R}}+\tilde{m}^2_{\mathrm{R}})\sum_{\beta_<}\frac{1}{2\omega^3_\beta\tilde{\omega}^2_{\mathrm{S}\beta}}, \quad (121)$$

we write the shifted frequency as

$$\tilde{\Omega}^{(\pm)}_{\mathrm{H}\alpha} = \tilde{\omega}_{\mathrm{H}\alpha} + \frac{\lambda_{\mathrm{B}}\hbar}{4\tilde{\omega}_{\mathrm{H}\alpha}}\left[J_0 + \frac{(m^2_{\mathrm{R}}+\tilde{m}^2_{\mathrm{R}})}{2}J_{-1} + \frac{(m^2_{\mathrm{R}}+\tilde{m}^2_{\mathrm{R}})^2}{2}\left\{\sum_{\beta_>}\frac{1}{2\omega^3_\beta\tilde{\omega}^2_{\mathrm{H}\beta}} - \sum_{\beta_<}\frac{1}{2\omega^3_\beta\tilde{\omega}^2_{\mathrm{S}\beta}}\right\} + \frac{1}{8\omega_\alpha}\left(1 \pm \frac{\omega_\alpha}{\tilde{\omega}_{\mathrm{H}\alpha}}\right)^2\right]$$
$$\tilde{\Omega}^{(\pm)}_{\mathrm{S}\alpha} = \tilde{\omega}_{\mathrm{S}\alpha} - \frac{\lambda_{\mathrm{B}}\hbar}{4\tilde{\omega}_{\mathrm{S}\alpha}}\left[J_0 + \frac{(m^2_{\mathrm{R}}+\tilde{m}^2_{\mathrm{R}})}{2}J_{-1} + \frac{(m^2_{\mathrm{R}}+\tilde{m}^2_{\mathrm{R}})^2}{2}\left\{\sum_{\beta_>}\frac{1}{2\omega^3_\beta\tilde{\omega}^2_{\mathrm{H}\beta}} - \sum_{\beta_<}\frac{1}{2\omega^3_\beta\tilde{\omega}^2_{\mathrm{S}\beta}}\right\} + \frac{1}{8\omega_\alpha}\left(1 \pm i\frac{\omega_\alpha}{\tilde{\omega}_{\mathrm{S}\alpha}}\right)^2\right]. \quad (122)$$

The first two terms of the MSPT frequency (122)

$$\tilde{\Omega}^{(\pm)}_{\alpha} \approx \tilde{\omega}_\alpha \pm \frac{\lambda_{\mathrm{B}}\hbar}{4\tilde{\omega}_\alpha}J_0 \quad (123)$$

gives, to the first order of $\lambda_{\mathrm{B}}\hbar$, the renormalized frequency

$$\tilde{\Omega}^2_{\mathrm{R}\alpha} \approx \tilde{\omega}^2_\alpha \pm \frac{\lambda_{\mathrm{B}}\hbar}{2}J_0. \quad (124)$$

Finally, the lowest order solutions which match the solutions (96) before phase transition and satisfy Eq. (27), are given by

$$\tilde{\varphi}_{\mathrm{H}\alpha} = \frac{1}{\sqrt{2\Omega^{(-)}_\alpha}}\left(\frac{\tilde{\Omega}^{(-)}_{\mathrm{H}\alpha} - \Omega^{(-)}_\alpha}{\tilde{\Omega}^{(+)}_{\mathrm{H}\alpha} + \tilde{\Omega}^{(-)}_{\mathrm{H}\alpha}}\right)e^{i\tilde{\Omega}^{(+)}_{\mathrm{H}\alpha}t} + \frac{1}{\sqrt{2\Omega^{(-)}_\alpha}}\left(\frac{\tilde{\Omega}^{(+)}_{\mathrm{H}\alpha} - \Omega^{(-)}_\alpha}{\tilde{\Omega}^{(+)}_{\mathrm{H}\alpha} + \tilde{\Omega}^{(-)}_{\mathrm{H}\alpha}}\right)e^{-i\tilde{\Omega}^{(-)}_{\mathrm{H}\alpha}t},$$
$$\tilde{\varphi}_{\mathrm{S}\alpha} = \frac{1}{\sqrt{2\Omega^{(-)}_\alpha}}\left(\frac{\tilde{\Omega}^{(-)}_{\mathrm{S}\alpha} - i\Omega^{(-)}_\alpha}{\tilde{\Omega}^{(+)}_{\mathrm{S}\alpha} + \tilde{\Omega}^{(-)}_{\mathrm{S}\alpha}}\right)e^{\tilde{\Omega}^{(+)}_{\mathrm{S}\alpha}t} + \frac{1}{\sqrt{2\Omega^{(-)}_\alpha}}\left(\frac{\tilde{\Omega}^{(+)}_{\mathrm{S}\alpha} + i\Omega^{(-)}_\alpha}{\tilde{\Omega}^{(+)}_{\mathrm{S}\alpha} + \tilde{\Omega}^{(-)}_{\mathrm{S}\alpha}}\right)e^{\tilde{\Omega}^{(-)}_{\mathrm{S}\alpha}t} \quad (125)$$

It is shown that the MSPT method, when applied to the mean field equation (81), yields the correct renormalized frequency even after the phase transition.

VI. CONCLUSION

In summary, we have studied certain aspects of the renormalization of quantum fields evolving out of equilibrium using the recently developed Liouville-von Neumann method [21]. The multiple scale perturbation theory (MSPT) provides a powerful and useful tool in finding an analytic approximate solution to nonlinear field equations. The mean-field equations for the auxiliary variables for the Gaussian state, which are equivalent to the vacuum expectation of the Hartree-Fock equation, are analyzed using the MSPT. As a simple quantum mechanical model, two coupled quartic oscillators are studied. In particular, the model with one unstable quartic oscillator, mimicking a soft mode of a nonequilibrium field, sheds some light on the effect of nonlinear mode coupling between the soft and the hard modes in the nonequilibrium evolution of phase transitions [31]. The MSPT method is employed to the Φ^4-theory before the phase transition and its lowest order solutions to the mean-field equations involve the correct renormalized frequency consistent with the Gaussian effective potential [23]. Also the MSPT method turns out to be a powerful and practical tool for the renormalization of nonequilibrium quantum fields such as they appear in phase transitions.

Although we have used Eq. (23) in order to express the modified frequencies in terms of the finite renormalized parameters, there is a caveat which is necessary to keep in mind. The renormalized mass is obtained from the quantum fluctuations of the Gaussian state around $\phi_c = 0$ for the static and stable Φ^4-theory. When the symmetry is spontaneously broken, $\phi_c = 0$ is no longer the true vacuum state. However, the justification of Eq. (23) in obtaining renormalized parameters, is solely dependent on the same Gaussian approximation whose quantum state is the time-dependent Gaussian state around $\phi_c = 0$ throughout the phase transition. This effectively amounts to studying the nonequilibrium field dynamics during the rolling down of the field and before the fluctuations start probing the true vacuum state. As far as the soft modes are concerned, they just see an inverted potential. Therefore, the lowest order solution of the MSPT method exponentially grows indefinitely and begins to oscillate slowly. The imaginary part of the solution, leading to the oscillation, determines the width of the Gaussian state at later times, roughly covering the true vacuum state. The exponential growth of the lowest order solution is terminated by the first order correction of the MSPT method when the field begins to sample around the true vacuum state [31].

In this paper we confined our attention to the Gaussian approximation using the LvN formalism (which is equivalent to the Hartree-Fock or mean-field method) to address issues related to renormalization of a spontaneously broken Φ^4-theory. We now point out some of the limitations of the Hartree-Fock method since an understanding of those limitations is crucial in developing techniques for more accurately describing nonequilibrium dynamics of phase transitions. As has been recognized in the literature, the Hartree-Fock method (sometimes called the *collisionless* Hartree-Fock method) neglects the effects of scattering. It also cannot account for late time thermalization of the system since it takes into account the effect of interactions only through a mean field. It is based on the assumption that scattering effects do not play a very crucial role in the early stages of phase transition dynamics. However, recent studies [32] indicate that scattering effects may be crucial to understanding the dynamical evolution of the system far from equilibrium. Similarly, in the Liouville-von Neumann method, the quartic part $\hat{H}_{\rm P}$, which is a perturbation to the quadratic (Gaussian) part $\hat{H}_{\rm G}$ and is negligible before the phase transition, grows comparable to $\hat{H}_{\rm G}$ and significantly contributes to the Gaussian state [33]. These terms may correspond to the next-to-leading order contribution to the effective action of Ref. [32]. The study of non-equilibrium phase transition dynamics beyond the Gaussian approximation is under investigation.

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